Two-dimensional Black Holes in a Higher Derivative Gravity and Matrix Model

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Abstract

We construct perturbatively a class of charged black hole solutions in type 0A string theory with higher derivative terms. They have extremal limit, where the solution interpolates smoothly between near horizon AdS_2 geometry and the asymptotic linear dilaton geometry. We compute the free energy and the entropy of those solution using various methods. In particular, we show that there is no correction in the leading term of the free energy in the large charge limit. This supports the duality of the type 0A strings on the extremal black hole and the 0A matrix model in which the tree level free energy is exact without any α' corrections.

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1 Introduction

Black holes are interesting objects in gravity and string theory, which may be awaiting the complete understanding on the quantum nature of gravity. Since black holes have a large curvature value region near the singularity, the non-perturbative formulation of gravity theory or its non-perturbative stringy generalization may be needed to understand the full quantum nature of black holes. Though full non-perturbative descriptions of M/string theories are not available yet, there are several interesting toy string models whose nonperturbative descriptions are known. One such class of toy models is the one of matrix models which correspond to the non-perturbative formulation of noncritical string theories. Specifically, various two dimensional string theories on the flat background can be reformulated in terms of matrix models. In some cases, the dual matrix models are believed to be a complete non-perturbative formulation of the corresponding noncritical string theories. Then, one may naturally ask whether there are black hole solutions in type 0A string theory, and if there are, how they can be incorporated and understood in the context of dual via matrix models.

It was proposed in [1] that the type 0A matrix model with $\mu = 0$ and non-zero RR fluxes $q_+ = q_- = q$ is dual to 0A string theory on the extremal black hole [2]. Later on, the type 0A matrix model was generalized [3] to incorporate two kinds of RR fluxes and found that it depends only on the combined flux[1] $Q = q_+ + q_-$. Since the generalized matrix model is the same as the matrix model with just one kind of RR flux and there is no evidence for the existence of black holes in the type 0A matrix model side, it was argued that the matrix model is not related to the 0A strings on black holes [3]. Furthermore, the curvature radius of the black hole solution is the order of string length scale, independent of charge, and therefore the low energy gravity can not be trusted and it is not clear whether the black hole exists at all. Nevertheless, there are some pursuits of matching between the matrix model and type 0A strings on black holes or AdS space [4][5][6] with partial success.

In this paper we will try to extend these efforts by including lowest order α' corrections in the low energy effective gravity for type 0A string theory. As we mentioned, the curvature radius of the black hole is the order of string length scale, and therefore higher derivative terms should be taken into account. One of the motivation of this work is to determine how the behavior of the black hole geometry is modified under the higher derivative correction.

We find the perturbative evidence of the existence of charged black holes even with the higher derivative terms in the type 0A string theory, which interpolate smoothly between the near horizon geometry and the asymptotic geometry. In the case of the extremal black hole, the near horizon geometry is AdS_2, as usual. This is in contrast to the four dimensional cases where it is not easy to find the interpolating solutions. We compute the exact entropy of this extremal black hole using Sen's formalism and find the condition on the coefficient a of the higher derivative term in order to have the extremal black hole solution. We also compute the free energy of the non-extremal black holes using Euclidean action approach and Wald's Noether charge method. In the extremal limit, the leading term, in the large charge limit, of the free energy turns out to be unaffected under the α'-correction. This agrees with the result from the matrix model, which supports the duality of

[1] There is an additional term which depends on the difference between RR fluxes, but it was irrelevant to arguments for black holes [3].

those two models.

The organization of the paper is as follows. In section 2, we briefly review some relevant aspects on the type 0A string theory and its dual 0A matrix model. We also review the black hole solutions in type 0A string theory. In section 3, we construct charged black hole solutions in the presence of higher derivative terms in the metric. We construct solutions, perturbatively in the coefficient a, and find the black hole geometries which interpolate near horizon region and asymptotic region, which is linear dilaton geometry. In section 4, we compute the free energy and the entropy of the charged black hole solutions, for both extremal and non-extremal cases using various approaches. The computation supports the duality between the 0A matrix model and the 0A strings on the extremal black hole geometry. In section 5, we draw some conclusions. In appendix A, we review some relevant results in the Noether charge method and in section B, we derive a generic relation between the temperature dependence of the radius of the time-like Killing horizon for a non-extremal black hole and the curvature radius of the near-horizon geometry of the corresponding extremal black hole.

2 Black holes in the 0A string theory and the 0A matrix model

The nonchiral projection of two dimensional fermionic string theories gives rise to two type 0 string theories, so called type 0A and type 0B. Only NS-NS and RR sectors survive under the projection, and thus the type 0 theories contain bosonic fields only. In type 0A string theory, the NS-NS sector contains a graviton, a dilaton and a tachyon, while the RR sector includes two one-form gauge fields [7][8]. It was known that the low energy effective theory of the 0A strings admits charged black hole solutions [2]. Based on the computation of the free energy, it was argued that the 0A string theory on those black holes is dual to the 0A matrix model [1][9][10]. In the first part of this section, we review the low energy effective theory of 0A strings and its black hole solutions. In the second part, we give some relevant features in the 0A matrix model for the comparison with black hole side results.

2.1 Black holes at the lowest order in α' : Review

The low energy effective action of type 0A string theory at the lowest order in α' is given by [7]

$$\begin{aligned} I_0 \;=\; \int d^2x\sqrt{-g}\,\Big[\frac{1}{2\kappa^2}\,e^{-2\Phi}\Big(R+4\nabla_\mu\Phi\nabla^\mu\Phi+\frac{8}{\alpha'}-f_1(T)(\nabla T)^2+f_2(T)\Big) \\ -\frac{2\pi\alpha'}{4}f_3(T)(F^+)^2-\frac{2\pi\alpha'}{4}f_3(-T)(F^-)^2-q_+F^+-q_-F^-\Big]\,, \qquad (1) \end{aligned}$$

where $F_\pm$ denote field strengths of two RR gauge fields and $q_\pm$ denote the corresponding charges, respectively. The theory admits the following linear dilaton geometry as a vacuum solution:

$$\begin{aligned} \Phi &= -k\varphi\,, \\ ds^2 &= -dt^2+d\varphi^2\,, \qquad (2) \end{aligned}$$

where all other fields vanish. It is also known that there exist charged black hole solutions in the model [1][2]. When the tachyon field T is turned off, RR gauge fields can be easily solved as

$$F^{+}_{01} = F^{-}_{01} = \frac{q}{2\pi\alpha'}\,, \qquad T = 0\ , \tag{3}$$

which corresponds to the configuration of the background D0-branes with charges given by $q_{\pm} = q\,$.

One may use this to integrate out RR gauge fields, and obtain the action of the form

$$I'_0 = \int d^2x\sqrt{-g}\left[\frac{1}{2\kappa^2}\; e^{-2\Phi}\Big(R + 4\nabla_\mu\Phi\nabla^\mu\Phi + 4k^2\Big) + \Lambda\right]. \tag{4}$$

Here we denoted the original cosmological constant as $k^2 = 2/\alpha'$ and new effective cosmological constant, coming from the gauge field contributions, as $\Lambda = -q^2/(2\pi\alpha')$. Therefore the low energy effective theory of type 0A string theory reduces to the two dimensional dilaton gravity with two kinds of cosmological constants, one of which is related to the charges of RR gauge fields. The theory admits the charged black hole solutions in which the dilaton is taken to be proportional to the spatial coordinate φ,

$$\Phi_0 = -k\varphi \tag{5}$$

while the metric is of the form

$$ds^2 = -l_0(\varphi)dt^2 + \frac{d\varphi^2}{l_0(\varphi)}\,, \tag{6}$$

with the factor $l_0(\phi)$ given by[2]

$$l_0(\varphi) = 1 - e^{-2k\varphi}\,\frac{1}{2k}\Big(M_0 - \Lambda\varphi\Big)\,. \tag{7}$$

As will be clear, M_0 may be regarded as a mass of the black hole. One can also obtain the extremal black hole solution where the position of the horizon and the mass are given in terms of the charge as

$$e^{2k\varphi_{ex}} = -\frac{\Lambda}{4k^2}\,, \qquad M_{ex} = \Lambda\,\varphi_{ex} + 2k\,e^{2k\varphi_{ex}} = -\frac{\Lambda}{2k}\Big[1 - \ln\Big(-\frac{\Lambda}{4k^2}\Big)\Big]\,. \tag{8}$$

The near horizon geometry of the extremal black hole becomes AdS_2.

2.2 The free energy in the 0A matrix model

In this section we give the expression of free energy in the 0A matrix model [7][3]. The free energy, $F \equiv \ln Z$, of the 0A matrix model with the Fermi energy level μ and Ramond-Ramond(RR) flux q compactified at the radius R (or at the temperature $1/(2\pi T) = \beta/(2\pi)$) is given by

$$\frac{\partial^3 F_{0A}}{\partial\mu^3} = \Big(\frac{\alpha'}{2}\Big)\,2R\ \mathrm{Im}\Bigg[\int_0^\infty dt\frac{t/2}{\sinh(t/2)}\frac{\sqrt{\frac{\alpha'}{2}}t/2R}{\sinh(\sqrt{\frac{\alpha'}{2}}t/2R)}\; e^{-i\sqrt{\frac{\alpha'}{2}}\mu t - \frac{1}{2}qt}\Bigg]\,. \tag{9}$$

[2]We set $2\kappa^2 = 1$ from now on.

In this integral form non-perturbative effects for R and α' are included. Since it is enough for us to consider the perturbative effects, we perform the series expansion on $x/\sinh x$ and integrate term by term. This gives us the perturbatively expanded form as

$$\begin{aligned}\frac{\partial^3 F_{0A}}{\partial\mu^3} &= \Big(\frac{\alpha'}{2}\Big) 2R\ \mathrm{Re}\Bigg[\sum_{m=0}^{\infty}(-1)^{m+1}(2m)!\Big(\mu\sqrt{\tfrac{\alpha'}{2}}-i\frac{q}{2}\Big)^{-2m-1}\times \\ &\qquad \sum_{n=0}^{m}\frac{|1-2^{1-2(m-n)}||1-2^{1-2n}||B_{2(m-n)}||B_{2n}|}{[2(m-n)]!(2n)!}\ \Big(\frac{\sqrt{\alpha'/2}}{R}\Big)^{2n}\Bigg].\end{aligned} \tag{10}$$

Note that index m corresponds to the genus expansion in the 0A string theory as $\Big(\mu\sqrt{\frac{\alpha'}{2}}-i\frac{q}{2}\Big)^{-1}$ in the matrix model plays the role of the string coupling g_s in the type 0A theory. On the other hand, the summation over n corresponds to the α'-expansion.

Since the thermodynamic free energy, $\mathcal{F}_{0A}$ is defined by

$$F_{0A}\equiv -\beta\mathcal{F}_{0A}\,, \tag{11}$$

we obtain

$$\begin{aligned}\mathcal{F}_{0A} &= -\sqrt{\frac{2}{\alpha'}}\frac{1}{\pi}\ \mathrm{Re}\Bigg[-\frac{1}{2}\Big(\mu\sqrt{\tfrac{\alpha'}{2}}-i\frac{q}{2}\Big)^2\ln\Big(\mu\sqrt{\tfrac{\alpha'}{2}}-i\frac{q}{2}\Big)+\frac{1}{24}\Big\{1+\Big(\frac{\sqrt{\alpha'/2}}{R}\Big)^2\Big\}\ln\Big(\mu\sqrt{\tfrac{\alpha'}{2}}-i\frac{q}{2}\Big) \\ &\qquad +\sum_{m=2}^{\infty}(-1)^{m-2}(2m-3)!\Big(\mu\sqrt{\tfrac{\alpha'}{2}}-i\frac{q}{2}\Big)^{2-2m}\ F_m(R)\Bigg],\end{aligned} \tag{12}$$

where

$$F_m(R)\equiv\sum_{n=0}^{m}\frac{|1-2^{1-2(m-n)}||1-2^{1-2n}||B_{2(m-n)}||B_{2n}|}{[2(m-n)]!(2n)!}\ \Big(\frac{\sqrt{\alpha'/2}}{R}\Big)^{2n}. \tag{13}$$

To match the matrix model results with those of the type 0A string theory on the two dimensional extremal black hole, we should take some appropriate limits in the above. First of all, we should take the infinite radius limit(*i.e.* zero temperature limit) with $\mu=0$. Furthermore, we should double the RR flux q to get the effects from two kinds of RR flux $q_+=q_-=q$. After all these limits are taken, we obtain the tree level part of the thermodynamic free energy as

$$\mathcal{F}^{tree}_{0A}=-\sqrt{\frac{2}{\alpha'}}\,\frac{1}{\pi}\ \mathrm{Re}\Bigg[-\frac{1}{2}\Big(\mu\sqrt{\tfrac{\alpha'}{2}}-iq\Big)^2\ln\Big(\mu\sqrt{\tfrac{\alpha'}{2}}-iq\Big)\Bigg]_{\mu=0}=-\sqrt{\frac{2}{\alpha'}}\,\frac{q^2}{4\pi}\ln q^2\,, \tag{14}$$

Note that we have suppressed the ambiguous quadratic terms on q. They are related to the divergent parts which exist in the free energy expressions and may be regularized by the explicit cutoff. One may also note that there is no further α' correction in the tree level free energy of the type 0A matrix model, which is not the case for higher genus ones. In the next section, we will consider higher derivative corrections to the low energy effective action of type 0A string theory and check that it gives the same free energy with the type 0A matrix model.

3 Black holes in the higher derivative type 0A gravity

Now we would like to include the higher derivative correction, i.e. higher order α' correction to the action given in the previous section. We restrict ourselves to the higher derivative terms in the NS-NS sector fields only[3]. The results from β function computation [11] tell us that we do not need to consider higher derivative terms in Φ. Furthermore, the unique higher derivative term for the metric in two dimensions, which appear in the next order in α' correction, is a R^2 term. Henceforth, it is enough to add the following correction term to the action (4):

$$I_1 = \frac{1}{2\kappa^2} \int d^2x\sqrt{-g}\, e^{-2\Phi}\Big(a\alpha' R^2\Big)\,, \tag{15}$$

where a is a certain dimensionless number which may be fixed by the computation of α' corrections in string theory.

3.1 Black hole solutions

The equations of motion of the metric and the dilaton are found to be

$$\begin{aligned} 0 &= R_{\mu\nu} + 2\nabla_\mu\nabla_\nu\Phi + g_{\mu\nu}\Big[2\nabla^2\Phi - 4(\nabla\Phi)^2 + 4k^2 + \frac{1}{2}e^{2\Phi}\Lambda\Big] - \frac{4a}{k^2}\, e^{2\Phi}\nabla_\mu\nabla_\nu(e^{-2\Phi}R)\,, \\ 0 &= R + 4\nabla^2\Phi - 4(\nabla\Phi)^2 + 4k^2 + \frac{2a}{k^2}\, R^2\,. \end{aligned} \tag{16}$$

The metric for the black hole solutions of the above equations of motion is chosen to be the Schwarzschild type as

$$ds^2 = -l(\varphi)dt^2 + \frac{d\varphi^2}{l(\varphi)}\,. \tag{17}$$

In general, we have three equations of motion in which only two of them are independent. After simple manipulation, the equations of motion for the metric component g_{tt} and dilaton Φ in the above Schwarzschild type metric are given by

$$\begin{aligned} 0 &= l'' - 6\,l'\,\Phi' - 4\,l\,\Phi'' + 8\,l\,(\Phi')^2 - 8k^2 - \Lambda\, e^{2\Phi} + \frac{4a}{k^2}\,(2\,l'\,l''\,\Phi' - l'\,l''')\,, \\ 0 &= l'' - 4\,l\,\Phi'' - 4\,l'\,\Phi' + 4\,l\,(\Phi')^2 - 4k^2 - \frac{2a}{k^2}\,(l'')^2\,, \end{aligned}$$

where $\Phi' = \frac{d\Phi}{d\varphi}$ and $l' = \frac{dl}{d\varphi}$. It is not easy to obtain a black hole solution analytically. Instead, we use pertubative approach to find out charged black hole solutions. We require that the solutions reduce to the well-known black hole solutions as $a \to 0$. We also require that the geometry approaches, asymptotically, to the linear dilaton geometry (2). The word 'perturbative' here means the perturbative expansion in terms of variable 'a' in the Eq. (15). Let us recall a is just a number fixed by α' correction and not a parameter we may arbitrarily vary. Furthermore there is no guarantee that it is small. Nevertheless we may still regard it as an adjustable variable and try to do a perturbative analysis. The partial justification of this approach is given by the comparison

[3] One may include the generic higher derivative terms in the RR sector as well.

of results from this approach with those from the exact one in the near horizon limit of extremal black hole.

First of all, the analysis of the asymptotic behavior of the dilaton and the metric leads to

$$\begin{aligned}\Phi(\varphi) &= -k\varphi + \mathcal{O}(e^{-4k\varphi}),\\ l(\varphi) &= 1 - e^{-2k\varphi}\frac{1}{2k}\Big(M - \Lambda\varphi\Big) + \mathcal{O}(e^{-4k\varphi}),\end{aligned} \tag{18}$$

where we denote 'M' as a mass of the black hole. As will be shown in later section, there is an ambiguity in defining the ADM mass in two dimensions, due to the divergencies. However the difference $M - M_{ex}$, where M_{ex} is the corresponding quantity in the extremal black hole with the same charge, is well defined as the energy above the extremal black hole, and thus it maybe justified to call M as the mass of the black hole. The mass of the black hole depends on a and may be written generically in the form

$$M = M_0 + aM_1 + a^2 M_2 + \cdots .$$

We introduce, for clarity, a dimensionless mass m and a dimensionless cosmological constant λ as

$$m = \frac{M}{k}, \qquad \lambda = \frac{\Lambda}{k^2}.$$

Through the perturbative expansion in a, we obtain the solutions of the equations of motion, up to second order in a, as

$$\Phi = -k\varphi - 16\,a^2\,e^{-4k\varphi}\Big(m_0 - \lambda k\varphi + \lambda\Big)^2 + \mathcal{O}(a^3), \tag{19}$$

$$l(\varphi) = l_0(\varphi) + a\,l_1(\varphi) + a^2 l_2(\varphi) + \mathcal{O}(a^3), \tag{20}$$

where

$$\begin{aligned}l_0(\varphi) &= 1 + e^{-2k\varphi}\Big[-\frac{1}{2}\Big(m_0 - \lambda k\varphi\Big)\Big],\\ l_1(\varphi) &= e^{-2k\varphi}\Big[-\frac{m_1}{2}\Big] + e^{-4k\varphi}\Big[\lambda^2 + 2\Big(m_0 - \lambda k\varphi\Big)^2\Big],\\ l_2(\varphi) &= e^{-2k\varphi}\Big[-\frac{m_2}{2}\Big] + e^{-4k\varphi}\Big[-96\lambda^2 + 4\Big(m_0 - \lambda k\varphi\Big)\Big(m_1 - 48\lambda\Big) - 128\Big(m_0 - \lambda k\varphi\Big)^2\Big]\\ &\quad + e^{-6k\varphi}\Big[-8\lambda^3 + 8\lambda^2\Big(m_0 - \lambda k\varphi\Big) + 48\lambda\Big(m_0 - \lambda k\varphi\Big)^2 + 16\Big(m_0 - \lambda k\varphi\Big)^3\Big].\end{aligned} \tag{21}$$

At first glance, one may think that m_i's are independent parameters describing the above perturbative solutions. However, this is not the case and the mass of a black hole is expressed in terms of just a single parameter m as can be seen from the fact that all the expressions of the given solution can be rewritten as the function of a single variable m up to the given order. In terms of mass parameter m, the solution can be rewritten as

$$\Phi = -k\varphi - 16\,a^2\,e^{-4k\varphi}\Big(m - \lambda k\varphi + \lambda\Big)^2 + \mathcal{O}(a^3), \tag{22}$$

$$
\begin{aligned}
l(\varphi) &= 1-\frac{e^{-2k\varphi}}{2}\Big(m-\lambda k\varphi\Big)+ae^{-4k\varphi}\Big[\lambda^2+2\Big(m-\lambda k\varphi\Big)^2\Big] \\
&\quad -32a^2e^{-4k\varphi}\Big[3\lambda^2+6\lambda\Big(m-\lambda k\varphi\Big)+4\Big(m-\lambda k\varphi\Big)^2\Big] \\
&\quad +8a^2e^{-6k\varphi}\Big[-\lambda^3+\lambda^2\Big(m-\lambda k\varphi\Big)+6\lambda\Big(m-\lambda k\varphi\Big)^2+2\Big(m-\lambda k\varphi\Big)^3\Big]+\mathcal{O}(a^3)\,.
\end{aligned}
\tag{23}
$$

This solution has several distinct features. First of all, the only combination which appear in the metric and the dilaton is of the form

$$
\lambda^p(m-\lambda k\varphi)^q e^{-2(p+q)k\varphi} \tag{24}
$$

Secondly, the coefficients of a^n in the dilaton Φ include, generically, terms proportional to $e^{-2lk\varphi}$, with $l=2,3,\cdots,n$, while the coefficients of a^n in the metric function $l(\varphi)$ contain terms proportional to $e^{-2lk\varphi}$, with $l=2,3,\cdots,n+1$. These generic features can be shown to be true in all orders of a from the recursive structure in the equations of motion. From these properties, we find an additional characteristic feature of the solution: the solution depends only on one combination $\tilde{m}$ of two parameters m and λ where

$$
\tilde{m}=\frac{m}{\lambda}-\frac{1}{2}\ln\Big(-\frac{\lambda}{4}\Big)\,.
$$

In order to see this, we introduce the shifted spatial coordinate $\tilde{\varphi}$:

$$
\tilde{\varphi}=\varphi-\frac{1}{2k}\ln\Big(-\frac{\lambda}{4}\Big)\,.
$$

Then the generic form displayed in (24) becomes $(\tilde{m}-k\tilde{\varphi})^q e^{-2(p+q)k\tilde{\varphi}}$, which depends only on the parameter $\tilde{m}$. This property will be useful later on.

3.2 Horizon and temperature

The horizon is determined in terms of given variables m and λ by the condition

$$
l(\varphi_H)=0\,. \tag{25}
$$

Using the perturbative solution of the function $l(\varphi)$ given in Eq.(23), up to a^2 order, one can reorganize the above relation between the mass parameter m and the horizon φ_H, which is more suitable to the perturbative determination of φ_H, as

$$
m-\lambda k\varphi_H = 2\,e^{2k\varphi_H}\Big[1+a\Big(8+\lambda^2\,e^{-4k\varphi_H}\Big)-8a^2\Big(32+24\lambda\,e^{-2k\varphi_H}+8\lambda^2\,e^{-4k\varphi_H}+\lambda^3\,e^{-6k\varphi_H}\Big)+\mathcal{O}(a^3)\Big]\,.
$$

The perturbative expression for the position of the horizon, φ_H, can be obtained by expanding it as a series in a as

$$
\varphi_H=\varphi_H^0+a\,\varphi_H^1+a^2\,\varphi_H^2+\mathcal{O}(a^3)\,, \tag{26}
$$

which gives us φ_H^i as functions of m_i $(i=0,1,2\cdots)$, and, in principle, φ_H as a function of m.

Temperature of these non-extremal black holes can be obtained from the condition of the absence of the conical singularity in the Euclideanized black hole geometry. In the case at hand, the temperature is given by the relation

$$T = \frac{1}{4\pi}\, l'(\varphi_H \ ; \ m, \lambda)\,.$$

Note that in terms of the shifted coordinate, it becomes $T = \frac{1}{4\pi}\, \partial_{\tilde{\varphi}} l(\tilde{\varphi}_H \ ; \ \tilde{m})$, which tells us that there is no λ-dependence in the relation between the Hawking temperature T and the position of the horizon in the shifted coordinate, $\tilde{\varphi}_H$. As we have seen, $\tilde{\varphi}_H$ is determined by the mass parameter $\tilde{m}$ ($\tilde{\varphi}_H = \tilde{\varphi}_H(\tilde{m})$) or vice versa ($\tilde{m} = \tilde{m}(\tilde{\varphi}_H)$) from the Eq. (25). After expressing the mass parameter $\tilde{m}$ in the metric function $l(\tilde{\varphi})$ in terms of the position of the horizon, $\tilde{\varphi}_H$, the above relation reduces to the one between the position of the horizon in the shifted coordinate and the Hawking temperature. Then, the formula can be inverted and $\tilde{\varphi}_H$ can be written in terms of T only. Up to a^2 order the perturbative relation is given by

$$\begin{aligned} k\tilde{\varphi}_H(T) &= k\varphi_H(T) - \frac{1}{2}\ln\Big(-\frac{\lambda}{4}\Big) \\ &= -\frac{1}{2}\ln\Big[1 - 2\pi\Big(\frac{T}{k}\Big) - 8a\Big\{1 - 4\pi\Big(\frac{T}{k}\Big) + 8\pi^2\Big(\frac{T}{k}\Big)^2\Big\} + \mathcal{O}(a^3)\Big]\,, \end{aligned} \tag{27}$$

One can take the extremal limit of these charged black hole solutions, which is characterized by the conditions

$$l(\varphi_{ex}) = 0\,, \qquad l'(\varphi_{ex}) = 0\,,$$

where the second condition tells that the Hawking temperature of the extremal black hole is zero. Up to a^2 order, the perturbative solution of the horizon position is given by

$$k\varphi_{ex} = \frac{1}{2}\ln\Big(-\frac{\lambda}{4}\Big) + 4a + 16a^2 + \mathcal{O}(a^3)\,, \tag{28}$$

while the parameter m_{ex} is given by

$$m_{ex} = -\frac{\lambda}{2}\Big[1 - \ln\Big(-\frac{\lambda}{4}\Big)\Big] - 12\lambda\, a + 16\lambda\, a^2 + \mathcal{O}(a^3)\,. \tag{29}$$

Note that the following combination of two parameters depends only on a :

$$\frac{1}{\lambda}\Big(m_{ex} - \lambda\, k\varphi_{ex}\Big) = -\frac{1}{2}\Big[1 + 32a + \mathcal{O}(a^3)\Big]\,. \tag{30}$$

This fact can be confirmed to hold for all orders in a by using the equations of motion.

4 Free Energy and Entropy of Black Holes

In this section we find the free energy and the entropy of our black hole solutions using various approaches. First of all we use Sen's entropy function formalism to obtain the full expression of entropy of the extremal black hole. Then we use Euclidean action formalism and Wald's Noether charge method to obtain the free energy and the entropy of non-extremal black holes.

4.1 Entropy of extremal black holes

For the extremal case, the exact expression for the entropy can be obtained for given Lagrangian by the entropy function formalism of Sen [12][13][14][15]. In this section, we use Sen's formalism to obtain the full expression of the entropy of the extremal black hole. In the next section, we will obtain the perturbative entropy of non-extremal black holes and compare it with the results in this section.

One can show that the following field configurations, which have the $SO(2,1)$ symmetry,

$$ds^2 = v\Big(-x^2 dt^2 + \frac{dx^2}{x^2}\Big)\,, \tag{31}$$

$$\Phi = u\,, \tag{32}$$

are the solutions of the equations of motion (16). These configurations may appear as the near-horizon limit of our extremal black holes. We found that it is the case, up to a^2 order, and we believe it holds, generically, all orders in a. This is supported by the fact that our perturbative solutions interpolate smoothly between the near horizon region, which is AdS_2 geometry, and the asymptotic region, which is linear dilaton geometry. It is in contrast to the higher-dimensional case [16], with higher derivative terms, where it is not easy to find the solution which interpolates smoothly between the near horizon and the asymptotic regions.

The entropy function is given by the value of the total Lagrangian at the horizon

$$F(v,u) = -2\pi v\Big[e^{-2u}\Big(-\frac{2}{v} + \frac{8a}{k^2}\frac{1}{v^2} + 4k^2\Big) + \lambda k^2\Big]\,. \tag{33}$$

The value of u and v are fixed by the extremum conditions:

$$\frac{\partial F}{\partial v} = 0\,, \qquad \frac{\partial F}{\partial u} = 0\,,$$

which lead to

$$v_* = \frac{1}{4k^2}\Big(1+\sqrt{1-32a}\Big)\,, \qquad e^{2u_*} = -\frac{2}{\lambda}\frac{1}{k^2 v_*}\Big(1-\frac{8a}{k^2 v_*}\Big)\,. \tag{34}$$

The entropy of the extremal black hole is the extremum value of the entropy function and is given by

$$S = F(v_*,u_*) = -\frac{\pi\lambda}{2}(1+\sqrt{1-32a})\,. \tag{35}$$

Note that a should satisfy $a \le \frac{1}{32}$ for the entropy to be real, which sets the condition for the model to have an extremal black hole solution. In order to compare with the perturbative results in the next section, we recover the original physical parameter $q^2 = -4\pi\lambda$ and expand in a and obtain

$$S = \frac{q^2}{4}\Big(1-8a-64a^2\Big) + \mathcal{O}(a^3)\,. \tag{36}$$

As will be shown, it is consistent with the results for the non-extremal black holes in the extremal limit using the Euclidean action method and the Noether charge one .

4.2 Non-extremal black holes : Euclidean action approach

There are several ways to get the entropy and finite temperature free energy for non-extremal black holes. We employ two well-known methods to calculate the finite temperature free energy for charged black holes, namely Euclidean action approach [17] and Wald's Noether charge method [18][19]. In this section, we get the free energy expression for non-extremal black holes by using the Euclideanized action. Before proceeding, let us recall that the naive ADM mass is divergent for a two dimensional black hole which asymptotes to the linear dilaton background. To regularize this divergence, one should introduce a cut-off. One way to remove the cut-off dependent terms is to subtract the value of the reference spacetime. One natural choice for the reference spacetime is the extremal black hole which has the same charge in the background as non extremal ones [20].

In the Euclidean action approach, we need the Gibbons-Hawking-York [21][17] boundary terms to get the well-defined variational problem and the correct free energy. Since we have a higher derivative correction term in the bulk Lagrangian, we should consider the corresponding correction in the boundary terms, as well. It turns out that this correction in the boundary terms has no effect to the free energy expression in the case at hand. The boundary terms, when the bulk Lagrangian is written in terms of an arbitrary function, $f(R)$, of the Ricci scalar R, can be easily found and the total action becomes

$$I_E = -\int_M \sqrt{g} f(R) - 2\int_{\partial M} \sqrt{h}\, f'(R)\, K\,, \qquad f'(R) \equiv \frac{d}{dR} f(R)\,, \tag{37}$$

where K is the trace of the second fundamental form of the boundary ∂M. The form of the boundary terms is valid even when $f(R)$ contain other fields, for instance, the dilaton field. In our case, $f(R)$ is given by

$$f(R) = e^{-2\Phi}\Big(R + \frac{2a}{k^2}R^2 + 4\nabla_\mu\Phi\nabla^\mu\Phi + 4k^2\Big) + \Lambda\,,$$

After plugging in our perturbative solution and taking the boundary at $\varphi = \bar{\varphi}$, the action of the bulk part contributes to

$$\int_{\varphi_H}^{\bar{\varphi}} d\varphi \sqrt{g} f(R) = 4k\, e^{2k\varphi} + 3\lambda k^2 \varphi + aI_1(\varphi) + a^2 I_2(\varphi)\Big|_{\varphi_H}^{\bar{\varphi}}\,, \tag{38}$$

where

$$I_1(\varphi) = e^{-2k\varphi} 4k\Big[\lambda^2 + 2(m-\lambda k\varphi)^2\Big]\,, \tag{39}$$

$$\begin{aligned} I_2(\varphi) &= e^{-2k\varphi}128k\Big[-5\lambda^2 - 9\lambda(m-\lambda k\varphi) - 5(m-\lambda k\varphi)^2\Big] \\ &\quad + e^{-4k\varphi}32k\Big[-\lambda^3 + 5\lambda^2(m-\lambda k\varphi) + 12\lambda(m-\lambda k\varphi)^2 + 4(m-\lambda k\varphi)^3\Big]\,. \end{aligned} \tag{40}$$

The boundary term in the Euclidean action is given by

$$\sqrt{h}\, f'(R)\, K\Big|_{\varphi=\bar{\varphi}} = \frac{1}{2}\, l'(\varphi)\, e^{-2\Phi}\Big(1 + a\frac{4}{k^2}\, R\Big)\Big|_{\varphi=\bar{\varphi}}\,. \tag{41}$$

After plugging in our perturbative solutions, the boundary part of the action becomes

$$I_{bd} = -\beta k\Big[m - \lambda k\bar{\varphi} + \frac{\lambda}{2} + \mathcal{O}(e^{-2k\bar{\varphi}})\Big]_{\bar{\varphi}\to\infty}. \tag{42}$$

The value of the total Euclidean action diverges as the cut-off $\bar{\varphi}$ goes to infinity. We remove these cut-off dependent terms by subtracting the action value of the reference extremal black hole with the same charges. The periodicity β_{ex} in the Euclidean time direction for the extremal black hole is chosen from the relation [22][23][9]

$$\beta_{ex}\sqrt{g^{ex}_{\tau\tau}(\bar{\varphi})} = \beta\sqrt{g_{\tau\tau}(\bar{\varphi})}\,, \tag{43}$$

so that the proper length of the reference geometry in the Euclidean time direction at the boundary coincides with the one of the non-extremal black hole geometry. Its asymptotic form becomes

$$1 - \frac{\beta_{ex}}{\beta} = \frac{M - M_{ex}}{4k}\,e^{-2k\bar{\varphi}} + \mathcal{O}(e^{-4k\bar{\varphi}})\,.$$

The regularized free energy is defined by

$$F_{BH}(\beta) = \frac{I_E(\beta)}{\beta} - \frac{I^{ex}_E(\beta_{ex})}{\beta_{ex}}\,,$$

which is determined to be, up to second order,

$$F_{BH}(T) = \Lambda\Big[\varphi_H(T) - \varphi_{ex}\Big]\,, \tag{44}$$

where $\varphi_H(T)$ and φ_{ex} are given in (27) and (28). Note that the temperature dependence of the free energy is entirely incorporated in the horizon position φ_H up to the second order in a. We conjecture the relation (44) is an exact one, holding for all orders in a. One of the reason that the relation (44) might be an exact one comes from the relation, derived in appendix B, between the temperature dependence of the horizon radius, or the position of the event horizon in two dimensions, of the non-extremal black hole and the curvature radius of the AdS_2 geometry, which is the near-horizon geometry of the corresponding extremal black hole. Indeed, from the relation (44), conjectured to hold all orders in a, we obtain the entropy of the extremal black hole as

$$S = -\frac{\partial F}{\partial T}\Big|_{T=0} = -\Lambda\frac{\partial\varphi_H(T)}{\partial T}\Big|_{T=0} = -2\pi\Lambda v_* = -\frac{\pi\Lambda}{2k^2}\Big(1 + \sqrt{1-32a}\Big)$$

which agrees with the exact expression (35) of the entropy of the extremal black hole.

Via a standard thermodynamic relation, we obtain the entropy of the non-extremal charged black holes as

$$\begin{aligned} S &= -\frac{\partial F}{\partial T} \\ &= -\frac{\pi\Lambda}{k^2}\Big(1 - 2\pi\Big(\frac{T}{k}\Big)\Big)^{-1}\Bigg[1 - 8a\bigg(\frac{1 - 8\pi(\frac{T}{k}) + 8\pi^2(\frac{T}{k})^2}{1 - 2\pi(\frac{T}{k})}\bigg) \\ &\quad -64a^2\bigg(\frac{1 - 12\pi(\frac{T}{k}) + 48\pi^2(\frac{T}{k})^2 - 96\pi^3(\frac{T}{k})^3 + 64\pi^4(\frac{T}{k})^4}{\big(1 - 2\pi(\frac{T}{k})\big)^2}\bigg)\Bigg] + \mathcal{O}(a^3)\,. \end{aligned} \tag{45}$$

and indeed, in the extremal limit, the entropy becomes

$$S = \frac{q^2}{4}\Big(1 - 8a - 64a^2\Big) + \mathcal{O}(a^3)\,, \tag{46}$$

which agrees with the result (36) from the Sen's formalism in the previous section.

4.3 Non-extremal black holes : Noether charge method

Now, we turn to the Noehter charge method pioneered by Wald [18][19][24] and obtain the same results as Euclideanized action method if we take the same prescriptions for removing the cut-off dependent terms. By using this method we may obtain more insights into the structure of free energy. As was done in Euclidean action approach, we may regularize the divergent expression by subtracting the reference spacetime which is given by the extremal black hole. For the basic materials on the Noether charge method relevant for our discussions, see appendix A.

The Noether charge for the action variation under the coordinate transformation by vector field ξ is

$$Q^{\mu\nu}_{\xi} = -2\Big[e^{-2\Phi}\Big(1 + \frac{4a}{k^2}R\,\Big)\nabla^{[\mu}\xi^{\nu]} + 2\xi^{[\mu}\nabla^{\nu]}\Big(e^{-2\Phi}(1 + \frac{4aR}{k^2})\Big)\Big]\,. \tag{47}$$

The relevant Killing vector field for the ADM-like mass and the free energy is the timelike one $\frac{\partial}{\partial t}$. Then, as shown in the appendix A, the energy is given by

$$\mathcal{E} \;=\; Q^{t\varphi} + B^{\varphi}\Big|_{\infty} \tag{48}$$

where

$$Q^{t\varphi} \;=\; e^{-2\Phi}\Big(1 + \frac{4a}{k^2}R\Big)\partial_{\varphi}l - 2l\partial_{\varphi}\Big(1 + \frac{4a}{k^2}R\Big)\,, \tag{49}$$

$$B^{\varphi} \;=\; e^{2k\varphi}\Big[2k\,l - \partial_{\varphi}l - 8k\,\Phi\Big]_{\infty}\,. \tag{50}$$

After plugging in the solution, the energy of the charged black hole is found to be

$$\mathcal{E} = M - \Lambda\varphi - 2k\,e^{2k\varphi} + 8k^2\varphi e^{2k\varphi} + \frac{16a\Lambda}{k}\Big|_{\infty}\,. \tag{51}$$

One may note that, as explained in the appendix A, there remain ambiguities in the choice of surface term Θ^{μ} and, accordingly, potential for currents $Q^{\mu\nu}$ in the Noether charge method. In the higher dimensional case, where ADM mass is well-defined and finite, those ambiguities do not affect the ADM mass and therefore do not represent any ambiguity. In two-dimensional case, the ADM-like mass is apparently divergent and, thus, should be taken with care. One may wonder whether those apparent ambiguities really lead to any real ambiguity in the ADM-like mass. It turns out that they affect only on the cut-off dependent part, which will be removed eventually, leaving the essential part,

$$M - \Lambda\varphi + \frac{16a}{k}\Lambda$$

invariant.

In the Noether charge method, the entropy is given by the value of Noether charge at the horizon and thus it becomes

$$S = \frac{1}{T}\, Q^{t\varphi}\Big|_{\varphi_H} = 4\pi\ e^{-2\Phi}\,(1 + \frac{4a}{k^2}\,R)\Big|_{\varphi_H}\,. \tag{52}$$

The entropy itself is finite and well-defined. It is totally free from the ambiguities in defining Noether charge mentioned above. Through the thermodynamic relation among the free energy, the energy and the entropy as

$$\mathcal{F}_{BH} = \mathcal{E} - TS\,, \tag{53}$$

we obtain the free energy F_{BH} with respect to the reference geometry:

$$\begin{aligned} F_{BH} &= \mathcal{F}_{BH} - \mathcal{F}^{ex}_{BH} \\ &= (M - M_{ex}) - T\Big[4\pi\ e^{-2\Phi}\Big(1 + \frac{4a}{k^2}R\Big)\Big]_{\varphi_H}\,. \end{aligned} \tag{54}$$

When our black hole solutions are inserted in the above equation, the same results as Euclidean action approach are obtained, up to the second order in a, as

$$F_{BH}(T) = \Lambda\Big[\varphi_H(T) - \varphi_{ex}\Big]\,. \tag{55}$$

So far we have used the extremal black holes as our reference geometry to remove the cut-off dependent terms in the mass of non-extremal black holes. This is the same prescription we used in the Euclidean action formalism where it seems to be the only natural choice to deal with cut-off dependent terms. As a result, the free energy for the extremal black hole is assigned to be zero. But in the Wald's Noether charge method, there seems to be more natural prescription in dealing with the cut-off dependent terms. We may just remove all the cut-off dependent terms in the mass formula for the non-extremal black hole. If we follow this prescription, the free energy becomes

$$F_{BH}(T) = (M + \frac{16a}{k}\Lambda) - T\Big[4\pi\ e^{-2\Phi}\Big(1 + \frac{4a}{k^2}R\Big)\Big]_{\varphi_H}\,. \tag{56}$$

In the extremal limit, the free energy is given by

$$F^{ex}_{BH} = (M_{ex} + \frac{16a}{k}\Lambda) = -\frac{1}{\sqrt{2\alpha'}}\frac{q^2}{4\pi}\ln q^2 + \mathcal{O}(q^2)\,. \tag{57}$$

In the leading term, i.e. $q^2 \ln q^2$, there is no α'-correction, which agrees with the result from the matrix model. There is inherently an overall factor 2 difference between the free energy of the extremal black hole and the matrix model. This may be resolved by rescaling the time coordinate of black hole.

5 Discussion

In this paper we constructed charged black hole solutions in the low energy effective theory of type 0A string theory with α' correction term. Though they are not exact forms, but are given only perturbatively in a, they interpolate smoothly between the near horizon geometry and the

asymptotic dilaton geometry, which strongly indicates the existence of the solutions. They also have smooth extremal limit whose near horizon geometry is AdS_2, which can be easily confirmed to be the exact solution of the equations of motion with the higher derivative term if $a \le 1/32$.

We computed the free energy and the entropy of the non-extremal black holes using Euclidean action and Wald's Noether charge approach. There are divergent or cut-off dependent terms in the computation of ADM-like mass in two dimensions due to the linear dilaton. We discussed the subtleties in removing those cut-off dependent terms. The leading term of free energy in the large charge limit is independent of the method of removing cut-off dependent terms and thus shown to be well defined. It was shown to be the same as the one from matrix model. This supports the duality between the 0A string theory and the 0A matrix model. Though the geometry itself is given only perturbatively, we found the exact expression of the entropy of the extremal black hole and also found an exact relation between the free energy of the non-extremal black hole and the the position of Killing horizon.

It is believed that the matrix model does not admit the black hole solution. On the other hand there are exact black hole solutions in the low energy effective theory of the dual type 0A string theory. One argument against the existence of black hole solutions is that the curvature radius is the order of string length scale, independent of the charge, and thus low energy effective gravity theory can not be trusted. We considered the possible higher derivative term in type 0A gravity and found the condition of the existence of the black hole solutions. By direct computations in string theory, one may get the value of a to confirm the (non-)existence of the 0A black holes.

It would be very interesting to include the higher derivative terms in one-form gauge fields and the combination of them with NS-NS fields. Since the 0A string theory is not supersymmetric, it is not clear how to incorporate those terms. Another interesting problem is whether the duality holds when we include the quantum correction in the 0A string theory due to the tachyon field. Works in this direction is under progress.

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Appendix A

In this appendix, we summarize some relevant results in Noether charge method for the entropy and the free energy by Wald [18][19] [24][25][26].

Under a general field variation $\delta\phi^a$, the variation of the Lagrangian is given by

$$\delta(\sqrt{-g}\mathcal{L}) = \sqrt{-g}E_a\delta\phi^a + \partial_\mu(\sqrt{-g}\Theta^\mu)\,,$$

where E_a denotes the equations of motion for various fields, ϕ^a, and Θ corresponds to the surface term. The Noether current of a vector field ξ for diffeomorphism invariance can be obtained as

$$J^\mu_\xi = \Theta^\mu_\xi - \mathcal{L}\,\xi^\mu\,. \tag{58}$$

The potential Q for the current is defined by

$$\nabla_\nu Q^{\mu\nu} = J^\mu\,. \tag{59}$$

Though there are ambiguities in the definition of the potential Q, Wald showed that the black hole entropy and the free energy can be written unambiguously in terms of Q as

$$S = \frac{1}{T}\int_H d\Sigma_{\mu\nu}\;Q^{\mu\nu}\,, \qquad F = \mathcal{E} - TS\,, \tag{60}$$

where H denotes bifurcate Killing horizon for Killing vector ξ and $\mathcal{E}$ is the ADM-like energy which can also be obtained by Noether charge method. More explicitly, it was shown that, whenever one can write

$$\delta\int_\infty d\Sigma_{\mu\nu}\Big(B^\mu\xi^\nu - B^\nu\xi^\mu\Big) = \int_\infty d\Sigma_{\mu\nu}\Big(\Theta^\mu\xi^\nu - \Theta^\nu\xi^\mu\Big)\,,$$

$\mathcal{E}$ can be defined by

$$\mathcal{E} = \int_\infty d\Sigma_{\mu\nu}\;\Big(Q^{\mu\nu} - (B^\mu\xi^\nu - B^\nu\xi^\mu)\Big)\,. \tag{61}$$

Since the ambiguities in $Q^{\mu\nu}$ may play some roles in our case, we explain them in some detail. One of them comes from the total derivative terms added in the Lagrangian, which do not change the equations of motion

$$\sqrt{-g}\mathcal{L} \longrightarrow \sqrt{-g}\mathcal{L} + \partial_\mu(\sqrt{-g}\boldsymbol{\mu}^\mu)\,.$$

This additional term modifies Θ as

$$\Theta^\mu \longrightarrow \Theta^\mu + \delta\boldsymbol{\mu}^\mu + \boldsymbol{\mu}^\mu\;\frac{1}{2}g^{\alpha\beta}\delta g_{\alpha\beta}\,.$$

Note that the transformation of the added term in Θ under the coordinate transformation (*i.e.* its Lie derivative) is given by

$$\delta_\xi\boldsymbol{\mu}^\mu + \boldsymbol{\mu}^\mu\frac{1}{2}g^{\alpha\beta}\delta_\xi g_{\alpha\beta} = \xi^\alpha\nabla_\alpha\boldsymbol{\mu}^\mu - \boldsymbol{\mu}^\alpha\nabla_\alpha\xi^\mu + \boldsymbol{\mu}^\mu\nabla_\alpha\xi^\alpha\,.$$

There is another ambiguity in Θ from its definition. Namely, one can modify Θ as

$$\Theta^\mu \longrightarrow \Theta^\mu + \nabla_\nu\mathbf{Y}^{\nu\mu}\,,$$

where $\mathbf{Y}^{\mu\nu}$ is an antisymmetric tensor. This leads to the modified form of the Noether current

$$J^\mu \longrightarrow J^\mu + \nabla_\nu\Big(\xi^\nu\boldsymbol{\mu}^\mu - \xi^\mu\boldsymbol{\mu}^\nu + \mathbf{Y}^{\nu\mu}\Big)\,.$$

The Noether charge itself has ambiguity in its definition, $J^\mu = \nabla_\nu Q^{\nu\mu}$, up to total derivative term

$$Q^{\nu\mu} \longrightarrow Q^{\nu\mu} + \nabla_\rho\mathbf{Z}^{\rho\nu\mu}\,,$$

where $\mathbf{Z}^{\rho\nu\mu}$ is a totally antisymmetric tensor. Therefore, these ambiguities can modify the Noether charge, $Q^{\mu\nu}$ as

$$Q^{\mu\nu} \longrightarrow Q^{\mu\nu} + \xi^\nu \boldsymbol{\mu}^\mu - \xi^\mu \boldsymbol{\mu}^\nu + \mathbf{Y}^{\nu\mu} + \nabla_\rho \mathbf{Z}^{\rho\mu\nu} \,. \tag{62}$$

For completeness, we give an example how the above various tensors are related to Wald's differential forms [18][19]. The tensor $\boldsymbol{\mu}^\mu$ is related to the component of the differential form $\mu_{\beta_1\cdots\beta_{D-1}}$ in D-dimensional spacetime case as

$$\boldsymbol{\mu}^\mu = \frac{1}{(D-1)!}\,\epsilon^{\mu\beta_1\cdots\beta_{D-1}}\,\mu_{\beta_1\cdots\beta_{D-1}}\,, \qquad \sqrt{-g}\epsilon^{01\cdots D-1} = -1\,.$$

Appendix B

In this appendix, we derive a somewhat generic relation between the temperature dependence of the radius of the time-like Killing horizon for a non-extremal black hole and the curvature radius of the near horizon geometry in the extremal limit.

For concreteness, let us take the metric as a Schwarzschild type as

$$ds^2 = -l(r)\,dt^2 + \frac{dr^2}{l(r)} + r^2 d\Omega_{D-2}^2\,.$$

Now, let us recall relations leading the horizon and temperature

$$l(r_H) = 0\,, \qquad T = \frac{1}{4\pi}\,l'(r_H)\,, \qquad\qquad l' \equiv \frac{d}{dr}l(r)\,.$$

If the radius of the horizon and the mass of non-extremal black holes are assumed to be written as an analytic function of temperature, which is not a valid assumption in the case of Schwarzschild black holes, the generic form of the expansions are

$$\begin{aligned} r_H(T, Q^a) &= r_0(Q^a) + f(Q^a)\,T + \mathcal{O}(T^2)\,, \\ m(T, Q^a) &= m_0(Q^a) + c(Q^a)\,T + \mathcal{O}(T^2)\,, \end{aligned}$$

where Q^a denote various charges in the given setup. Note that r_0 and m_0 are the radius of the horizon and the mass of the corresponding extremal black hole, respectively, and thus satisfy

$$l(r_0;\, m_0) = l'(r_0;\, m_0) = 0\,.$$

Then, the temperature relation, when $l'(r_H)$ is expanded near r_0, leads to

$$4\pi\,T = \Big[f(Q)\;l''(r_0\,;\,m_0, Q) + c(Q)\,\frac{\partial}{\partial m}\,l'(r_0\,;\,m(T), Q)\Big|_{T=0}\Big]T + \mathcal{O}(T^2)\,,$$

where there is no temperature independent term on the right hand side due to the extremal condition, $l'(r_0\,;\,m_0, Q) = 0$. On the other hand the horizon condition becomes

$$0 = l(r_H\,;\,m, Q) = c(Q)\,\frac{\partial}{\partial m}\;l(r_H\,;\,m, Q)\Big|_{T=0} T + \mathcal{O}(T^2)\,,$$

which gives the condition

$$c = 0 \qquad\qquad \text{as} \quad \frac{\partial}{\partial m}\, l \neq 0\,.$$

As a result, we get

$$\frac{\partial r_H(T)}{\partial T}\Big|_{T=0} = f(Q) = \frac{4\pi}{l''(r_0\,;\, m_0, Q)}\,, \tag{63}$$

which shows the relation between the temperature dependence of the horizon's radius of a non-extremal black hole and the curvature radius of the near horizon geometry in the corresponding extremal black hole, since the near horizon geometry in the extremal limit under the assumption $l''(r_0\,;\, m_0, Q) \neq 0$ is given by

$$ds^2_{near} = \frac{2}{l''(r_0\,;\, m_0, Q)}\Big[-x^2dt^2 + \frac{dx^2}{x^2}\Big] + r_0^2 d\Omega^2_{D-1}\,, \qquad x \equiv \frac{1}{2}\, l''(r_0\,;\, m_0, Q)\; r\,,$$

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